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# Singularities of plane curves which occur as singular sections of the bifurcation sets of the cuspoid catastrophes

F J Wright

Institut für Informationsverarbeitung, Universität Tübingen, Köstlinstrasse 6, D-7400 Tübingen 1, West Germany†

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**Abstract.** Plane curves specified parametrically by $y = s^p$, $x = s^q$, $p$ and $q$ integral, $p \geqslant q > 0$ are either graphs of monomials, or have a singularity at the origin which can be one of four types: cusp, bend, kink or end. These types are classified by $(p, q) = $ (even, odd) or by a winding number index. Singular sections of the cuspoid bifurcation sets, in which all but two of the control variables are set to zero, are either cusps, bends or one of four more-singular curves derived from the above by the addition of the tangent at the origin. An explicit classification is given of the curve types arising in codimension up to 7. Only in such sections can the equations of the bifurcation sets generally be written explicitly.

## 1. Introduction

Geometrical singularities of curves, surfaces etc take on new significance in catastrophe theory (Thom 1975, Zeeman 1977, Poston and Stewart 1978), in which they are the physical manifestations of multiple degeneracy in the state of a system.

A state of a complicated system, such as a biological system, may require a vast number of *state variables* $s_i$ to specify it. However, the system usually evolves in a space of small dimension, such as the space–time of our own existence. The variables in this evolution space, together with any parameters by which we can control the system, are called *control variables* $c_i$. In catastrophe theory the set of points $c$ in the control space $C$, at which the state of the system is degenerate, is called the *bifurcation set* $\mathscr{B}$—it divides the control space into non-degenerate regions. In the neighbourhood of a simple degeneracy, where just two states coalesce, $\mathscr{B}$ is smooth. In the neighbourhood of a multiple degeneracy $\mathscr{B}$ displays a geometrical singularity, whose type is characteristic of the form of degeneracy which produces it: for example, a triple degeneracy produces a cusp in $\mathscr{B}$. Much interest focuses upon the bifurcation set, because it represents an encapsulation in a low-dimension space of the essential geometry of the catastrophe (i.e. the degeneracy) which may actually occur in very high dimension.

A striking and familiar instance of this is optical caustics, perhaps most familiar as the rainbow, but also often seen as the bright focal lines under rippling water in a bath or swimming pool, or on the surface of a cup of tea (Berry and Upstill 1980 and references therein). The caustic is the bifurcation set, which is the physical manifestation of the configuration of the rays, or the shape of the wavefronts, themselves essentially

† Visiting fellow of the Alexander von Humboldt Foundation. Permanent address: Department of Applied Mathematics, Queen Mary College, London University, Mile End Road, London E1 4NS, England.

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invisible. A cusped caustic arises from the focusing of three rays, whereas on smooth parts of the caustic only two rays focus. Virtual caustics (defined as in the footnote on p 164 of Born and Wolf 1975) are also extremely common, although not so obvious. For example, the image one sees looking through a perfect pane of glass is a very small segment of a cusped virtual caustic (Wright 1981).

In elementary catastrophe theory the states $s$ ($\in S$, the *state space*) of the system are given as the stationary points of some function $\phi(s; c)$. When the parameters $c$ lie on the bifurcation set, two or more stationary points coalesce. The basis of catastrophe theory is Thom's theorem, which says that under suitable restrictions on the dimensions of $S$ and $C$, the behaviour of a typical $\phi$ is equivalent to that of one of a small finite set of *normal forms*.

The normal forms are distinguished, among other things, by the minimal dimension of $S$ (*corank*) and of $C$ (*codimension*), in which they can occur. In this paper we study the family of normal forms of corank 1, which is called the *cuspoids*. For these, the equivalence of $\phi$ referred to above is by smooth diffeomorphism, i.e. infinitely differentiable change of coordinates possessing an infinitely differentiable inverse. If $\dim S > 1$ other normal forms may also occur, and if $\dim C > 5$ it may be necessary to weaken the equivalence to homeomorphism for catastrophes other than cuspoids in order that the set of equivalence classes remain finite, i.e. it may be necessary to drop the differentiability requirement.

One of the most powerful aspects of Thom's theorem is that for a given $\dim C$, increasing $\dim S$ above some threshold does *not* increase the number of normal forms to which $\phi$ may be equivalent. Hence the cuspoids are relevant to systems with very large numbers of state variables.

The fact that $\phi$ is embedded in an equivalence class means that a particular $\phi$ is *structurally stable* under small perturbations. The cuspoid of codimension $K$ also occurs as a *stable singularity* of a general (non-gradient) map from $\mathbb{R}^K$ to $\mathbb{R}^K$. This is not true for catastrophes of higher corank, although these may also occur as stable singularities of maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ where $n > K$. For a clear explanation of this see Thorndike *et al* (1978) and Nye and Thorndike (1980). (For the mathematics see Golubitsky and Guillemin 1973).

There is clearly a problem of representation for point sets of high dimension. The best one can do is to accurately plot two-dimensional sections, or sketch three-dimensional sections. Generally such sections do not have simple equations, and their accurate plotting requires some numerical technique plus computer graphics. Woodcock and Poston (1974) have plotted a large number of such sections, in which the bifurcation set is represented as the envelope of a family of straight lines (for cuspoids) or curves (for umbilics). Upstill (Berry and Upstill 1980, Upstill 1979a, b) plots bifurcation sets of corank-two catastrophes by mapping, via $\nabla_s\phi = 0$, the zero contour of the Hessian determinant of $\phi(s; c)$ with respect to $s$ into $C$.

The sections of bifurcation sets which are usually plotted are the *coordinate sections* (despite the non-typical symmetry which they often show, as remarked upon by Nye and Thorndike (1980, p 6), specified by setting all but two of the control variables equal to constants. These sections show how the catastrophe *organises* catastrophes of lower codimension, but the *essential* geometry of a catastrophe only appears in the *singular* sections, i.e. those through $c = 0$.

The main motivation for this paper stems from the following. The sequence of cuspoid normal forms takes a very simple general form, which facilitates a systematic and exhaustive analytical study of properties of the cuspoids. The easiest way to find which catastrophe one is dealing with is often via the bifurcation set, especially if $\phi$ is complicated (e.g. Wright 1981). One can always write the equations of bifurcation sets in parametric form, with the state variables as parameters, but generally it is impossible (or too complicated to be useful) to write the equations explicitly; this is possible only in singular plane coordinate sections.

We shall find that in these special singular sections precisely six distinct types of curve occur. In the next two sections of the paper we discuss these curve types, and in § 4 we classify the singular coordinate sections of the cuspoid bifurcation sets. Structures emerge which are not obvious from previous studies. In §§ 5 and 6 we discuss the classification and some possible applications.

## 2. Singularities of a class of plane curves

The class of plane curves that we need to consider is those expressed parametrically as

$$y = s^p \qquad x = s^q \tag{1}$$

for all real $s$, where $p > 0$ and $q > 0$. Imposing, if necessary, the condition that $x$ and $y$ be positive (see below), we may eliminate $s$ to give the explicit equation

$$y = x^\alpha \qquad \text{where } \alpha = p/q > 0 \tag{2}$$

and all *real* roots are to be taken. Any singularity of (1) can only occur at the origin. By singularity we mean, of course, a point of non-analyticity of $y(x)$, which will be a branch point of (2) in the complex $x$ plane (or worse!). A point is singular if one or more derivatives fail to exist there, and this occurs here in two ways.

First let us consider equation (2), and ignore the fact that it was derived from (1). We exclude the trivial straight lines resulting from $p = q$, so without loss of generality we may take $p > q$, hence $\alpha > 1$ (otherwise swap $x$ and $y$). Then (2) is once differentiable and has zero slope at the origin, since

$$\mathrm{d}y/\mathrm{d}x = \alpha x^{\alpha-1} \qquad \text{and} \qquad \alpha - 1 > 0.$$

However,

$$\mathrm{d}^n y/\mathrm{d}x^n = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)x^{\alpha-n}$$

and if $\alpha - n < 0$ this diverges as $x \to 0$, unless one of the factors in parentheses is zero. Hence if $\alpha = p/q$ is non-integral, all $n$th derivatives with $n > \alpha$ diverge, giving a singularity (a branch point) at the origin.

The second form of singularity is more subtle: the curve actually stops at the origin, and only exists in the first quadrant. We shall call this an *end* singularity. This will occur for all irrational $\alpha$, because we are restricting ourselves to real variables—an irrational root of a negative real number cannot be real. Thus there is only one type of singularity for irrational $\alpha$, the *irrational end* singularity. It will look much like the rational end singularities shown in figure 1($d$). Such non-algebraic singularities cannot arise from catastrophe theory, and from now on we consider only *rational* $\alpha = p/q$, with $p > q$, resulting from $p$ and $q$ which are *natural numbers.*

The *rational end* singularity is not implicit in equation (2): it results only from equation (1), and is not *necessarily* a branch point of equation (2). It occurs when $p$ and $q$ are both even so that $y \geqslant 0$ and $x \geqslant 0$, because then $s \geqslant 0$ and $s \leqslant 0$ generate exactly the same curve in $(x, y)$: the two branches of the curve have coalesced exposing an endpoint at the origin. In all other cases the curve passes continuously through the origin, which is a singularity unless $\alpha = p/q$ is integral. Hence the origin can be regular only if $q$ is odd, since $p$ and $q$ both even produces an end singularity.

The general form of the curve is determined by the ranges of values which $x$ and $y$ may take. We distinguish six cases, of which four are singular. These are set out in table 1.

**Table 1.** Classification of the curve types. (The index is explained in § 3.)

| Index | $p$ | $q$ | $y$ range | $x$ range | $\alpha = p/q$ | Name | Symbol |
|---|---|---|---|---|---|---|---|
| 1 | odd | even | all | $x \geq 0$ | always fractional | cusp | $\mathrm{C}\alpha$ |
| $\frac{1}{2}$ | even | odd | $y \geq 0$ | all | fractional | bend | $\mathrm{B}\alpha$ |
| | | | | | even integer | even monomial | $\mathrm{M}\alpha$ |
| 0 | odd | odd | $y \geq 0$ | $x \geq 0$ | fractional | kink | $\mathrm{K}\alpha$ |
| | | | $y \leq 0$ | $x \leq 0$ | odd integer | odd monomial | $\mathrm{M}\alpha$ |
| $\frac{1}{4}$ | even | even | $y \geq 0$ | $x \geq 0$ | all | end | $\mathrm{E}\alpha$ |

We specify a curve by a symbol made up of a letter indicating the curve type, plus the value of $\alpha$. We take $\alpha$ in lowest form, since two curves of the same type with the same $\alpha$, but different $p$ and $q$, are the same. In fact, all types of curve except ends are completely specified by $\alpha$ in lowest rational or integral form, and the letter symbols are merely an aid to clarity. But an end may have the same lowest form $\alpha$ value as any other curve, so in this case some additional distinction is essential. For all types of curve except ends the ranges of $x$ and $y$ are implicit in equation (2) with lowest form $\alpha$ value. This is because these curves are algebraic, but the end is *semi-algebraic* being specified by the equality $y = x^{\alpha}$ *plus the inequalities* $y \geq 0, x \geq 0$. These are the conditions, mentioned at the beginning of this section, which must be imposed when $s$ is eliminated if $p$ and $q$ are both even.

Figure 1 shows a selection of the curves described in table 1 for small values of $p$ and $q$, including the non-singular quadratic and cubic monomials M2 and M3. The curves were accurately computer plotted from equation (1), and should make the reason for the choice of names clear (cusp is, of course, standard).

## 3. The singularity types

We have distinguished four equivalence classes of 'rational singularities'. The members of each class, labelled by their $p$, $q$ values, clearly have the same *form* in some sense, but in just what sense is quite subtle (see also Berry 1980, § 3.2). Simple topological equivalence, or homeomorphism, only distinguishes the end singularity—the others are all homeomorphic. If we try to strengthen the equivalence by making it infinitely differentiable then *none* of the curves are equivalent. Homeomorphism is too weak and diffeomorphism is too strong: we need something in-between. Of course, we have the $(p, q) =$ (even, odd) classification, but this depends on a particular coordinate system. We would like to have a coordinate-independent classification.

One possibility is an index or winding-number classification derived from the rotation of the tangent as the curve is traversed. (For references to winding-number classifications of some completely different singularities see e.g. Berry 1980, §§ 2.1,

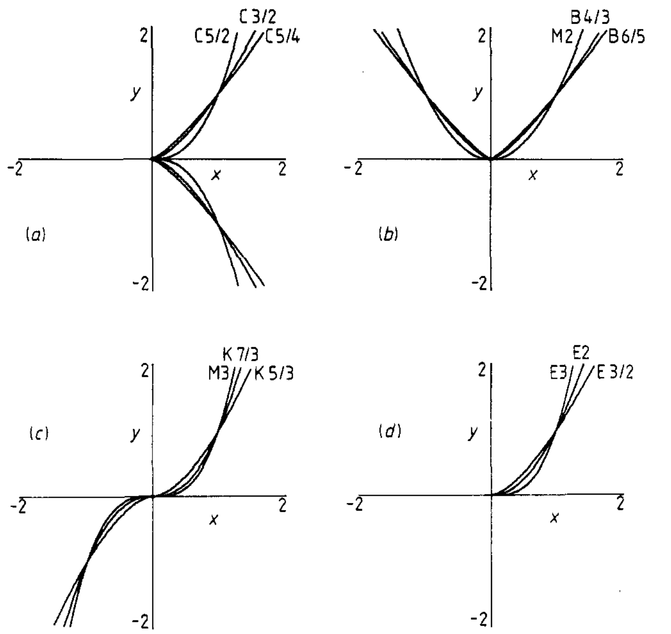


**Figure 1.** Some typical curves: (*a*) cusps, (*b*) bends and quadratic, (*c*) kinks and cubic, (*d*) ends.

3.3.) Label points along the curve by a parameter $s$, such as arc length, which increases monotonically with distance along the curve from $-\infty$ as $x$ or $y \to -\infty$, through 0 at the origin, to $+\infty$ as $x$ and $y \to +\infty$. Let $\theta(s)$ be the angle from (say) the $x$ axis in the positive direction (increasing $x$) to the tangent in the positive direction (increasing $s$). The slope of every curve tends to zero at the origin and infinity as $x$ and $y$ tend to $\pm\infty$, hence $|\theta(0)| = 0$, and $|\theta(\pm\infty)| = \frac{1}{2}\pi$. Let $\Delta\theta$ be the *total* change in $\theta$ along the curve. Then we define the index to be $I = |\Delta\theta/2\pi|$, where the modulus sign makes $I$ independent of the direction of traversing the curve. From figure 1, every curve has $\theta(0+) = 0$ and $\theta(+\infty) = \frac{1}{2}\pi$. Hence $E$ has $\Delta\theta = \frac{1}{2}\pi$ and $I = \frac{1}{4}$. $K$, $B$ and $M$ have $\theta(0-) = 0$, so $\theta(s)$ passes continuously through 0 at $s = 0$. $K$ and $M$(odd) have $\theta(-\infty) = +\frac{1}{2}\pi$, so $\Delta\theta = \frac{1}{2}\pi - \frac{1}{2}\pi$ and $I = 0$. $B$ and $M$(even) have $\theta(-\infty) = -\frac{1}{2}\pi$, so $\Delta\theta = \frac{1}{2}\pi + \frac{1}{2}\pi$ and $I = \frac{1}{2}$.

The cusp is a little tricky. $\theta(-\infty) = +\frac{1}{2}\pi$ but $\theta(0-) = \pi$, so $\theta(s)$ is discontinuous at the origin. Since angles are only defined modulo $2\pi$, we have an ambiguity whether to add $\pm\pi$ as $s$ passes through 0. Let us insist that $\theta(s)$ vary monotonically through a discontinuity, so that we add $\pi$ as $s$ increases. Then $\Delta\theta = \frac{1}{2}\pi + \pi + \frac{1}{2}\pi$ and $I = 1$. This is equivalent to regarding a cusp as the limit as a loop shrinks to a point.

The index classification is shown in the first column of table 1. Note that the two cases in which the curve crosses its tangent have integral index, otherwise the index is fractional. The index classification is coordinate independent, but is a global classification and locally cannot be applied exactly. The $(p, q)$ classification requires the particular choice of coordinates used in § 2, but applies locally (in keeping with the spirit of catastrophe theory).

We shall show that, in the cuspoid bifurcation sets, cusps and bends occur alone, and all the types of curve we have been considering occur in a closely related form involving a more complicated singularity. This is produced by adding the tangent at the origin, i.e.

each new curve is the union of the old one and the $x$ axis. *Every* such curve-plus-tangent has a singularity at the origin, even when the basic curve is a perfectly regular monomial. Every one of the new curves contains one or two copies of a special type of cusp, in which one limb is a straight line, and these special cusps are not diffeomorphic to any $C\alpha$.

Once the tangent is added, there is little point in distinguishing bends and kinks from monomials, since the tangent makes them all singular. Hence we have eight singular curves: C, CT, B, BT, K, KT, E, ET, where T stands for the added tangent, plus the regular monomials. C, CT, B, BT, KT, ET occur in singular sections of cuspoid bifurcation sets.

## 4. Singular coordinate sections of the cuspoid bifurcation sets

The cuspoid catastrophe of codimension $K, A_{K+1}$ in Arnol'd's (1975) notation, is generated by the normal form

$$\phi(s;c)=\frac{s^{K+2}}{K+2}+\sum_{n=1}^{K} c_n\frac{s^n}{n}$$

(using essentially the notation of Berry and Upstill 1980, Berry and Wright 1980). Its bifurcation set $\mathscr{B}$ is the solution in the control space $C$ of

$$\frac{\mathrm{d}\phi}{\mathrm{d}s}=s^{K+1}+\sum_{n=1}^{K} c_n s^{n-1}=0 \tag{3a}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}=(K+1)s^K+\sum_{n=2}^{K} c_n(n-1)s^{n-2}=0 \tag{3b}$$

where all variables are real. Equation ($3a$) specifies the states of the system as stationary points of $\phi(s;c)$ with respect to $s$, and equation ($3b$) is the condition that at least one state be degenerate, i.e. that ($3a$) has a multiple root.

As remarked earlier, it is not generally possible to eliminate $s$ from these equations. However, by making use of their linearity in the $c_n$, one can easily solve for the coordinates of the curve produced by *any plane* section, expressed parametrically in terms of $s$. Only in the singular coordinate sections, for which all but two of the $c_n$, say $c_i$ and $c_j$, are set equal to zero, can the equation of the curve be written explicitly. We call this a singular $(i,j)$ section, where without loss of generality we take $i>j$, hence $K\geqslant i>j\geqslant 1$. Then equations (3) become

$$s^{K+1}+c_i s^{i-1}+c_j s^{j-1}=0 \tag{4a}$$

$$(K+1)s^K+c_i(i-1)s^{i-2}+c_j(j-1)s^{j-2}=0. \tag{4b}$$

We can derive one generally valid solution of these equations by solving for $c_i$ and $c_j$, giving

$$\begin{aligned} c_i(i-j)&=-(K+2-j)s^{K+2-i}\\ c_j(i-j)&=(K+2-i)s^{K+2-j}. \end{aligned} \tag{5}$$

Let us define $p\equiv K+2-j$, $q\equiv K+2-i$. Then $K\geqslant i>j\geqslant 1$ implies that

$$2<p\leqslant K+1 \qquad 2\leqslant q<K+1 \qquad \text{and} \qquad p>q. \tag{6}$$

Equations (5) become

$$c_i(p-q)=-ps^q \qquad c_j(p-q)=qs^p$$

and defining $x\equiv-[(p-q)/p]c_i$, $y\equiv[(p-q)/q]c_j$, where the quantities in parentheses are always positive, we have precisely equations (1).

There is a second set of solutions of equations (4), which are easily overlooked, for which $s=0$. They lead to the tangent lines referred to earlier, and to more degenerate 'tangent planes'. We have to consider three cases. Consider first the case $j=1$ (hence $i>1$), when equations (4) become:

$$s^{K+1}+c_is^{i-1}+c_1=0 \tag{7a}$$

$$(K+1)s^K+c_i(i-1)s^{i-2}=0. \tag{7b}$$

$s=0$ satisfies equation (7$a$) if $c_1=0$. If $i=2$, the only solution of (7$b$) is then $c_2=0$, as already given by (5). But if $i>2$, $s=0$ satisfies (7$b$) for all $c_i$. Hence for $j=1$, $i>2$ there is a second solution $c_1=0$ for all $c_i$.

Consider now $j=2$ (hence $i>2$):

$$s^{K+1}+c_is^{i-1}+c_2s=0 \tag{8a}$$

$$(K+1)s^K+c_i(i-1)s^{i-2}+c_2=0. \tag{8b}$$

Now equation (8$a$) has a root at $s=0$ for all $c_i$ and $c_2$, which is simple if $c_2\neq0$. If $c_2=0$, $s=0$ also satisfies (8$b$) for all $c_i$, because $s=0$ is then a multiple root of (8$a$). Hence for $j=2$, $i>2$ there is a second solution $c_2=0$ for all $c_i$.

Finally consider $j\geqslant3$ (hence $i>3$):

$$s^{j-1}(s^{K-j+2}+c_is^{i-j}+c_j)=0 \tag{9a}$$

$$s^{j-2}[(K+1)s^{K-j+2}+c_i(i-j)s^{i-j}+c_j(j-1)]=0. \tag{9b}$$

$s=0$ is a solution of these equations for all $c_i$ and $c_j$, because of the factors in front of the brackets, i.e. $s=0$ is a $(j-1)$-fold root of (9$a$). This means that the whole $(i,j)$ plane is part of $\mathscr{B}$. The equations also have the solution $s=0$, $c_j=0$ (as for $j=2$), giving a tangent line on which the degeneracy at $s=0$ increases further.

In Arnol'd's notation catastrophes in a particular family, such as the cuspoids, are labelled by the maximum number of stationary points which coalesce. An $A_1$ point is a non-degenerate (Morse) stationary point, an $A_2$ point is a fold catastrophe point at which two stationary points coalesce, etc. Thus $A_1$ points do not contribute to $\mathscr{B}$, but any $A_n$ points with $n>1$ do. Using this notation we can summarise the singular coordinate sections of the cuspoid catastrophes as follows.

For all $i$ and $j$, almost all points $(c_i, c_j)$ give rise to one or more non-degenerate $A_1$ points at $s\neq0$. The roots of equation (9$a$) at $s\neq0$ can only be single or double. Hence, at points satisfying equation (5), other than the origin, pairs of $A_1$ points coalesce into $A_2$ points at $s\neq0$, so equation (5) gives a branch of $\mathscr{B}$ which is a fold line, or more than one fold (at different $s$) superposed in the $(i,j)$ plane.

For all $i$ and $j$, the $c_i$ axis ($c_j=0$), apart from the origin, is a line of $A_{i-1}$ points at $s=0$. For $i\geqslant3$ these are degenerate, producing the tangent line which forms a second branch of $\mathscr{B}$. For $j=2$ almost all points also give $A_1$ points at $s=0$, but these are non-degenerate and so do not contribute to $\mathscr{B}$.

For $j\geqslant3$ and all $i$ ($>j$) almost all points also give $A_{j-1}$ points at $s=0$. These are degenerate, so the whole $(i,j)$ plane is part of $\mathscr{B}$: it is a two-dimensional analogue of a tangent line. The $A_{j-1}$ points only unfold when the control parameters are moved out of the $(i, j)$ plane, so they do not produce any bifurcation within the plane.

At the origin, for all $i$ and $j$, these different branches and planes of $\mathscr{B}$ merge to give the main singularity, an $A_{K+1}$ point at $s = 0$. (The additional roots required result from the coalescence of *complex* conjugate pairs, which we have not considered.)

We are at last in a position to classify the singular coordinate sections of the cuspoid bifurcation sets. We classify them by curve type and codimension in table 2, indicating for each catastrophe which $(i, j)$ section displays a particular curve type. If the $i$ value is underlined it means that the $c_i$ axis forms part of $\mathscr{B}$, i.e. this section is of curve-plus-tangent type. If the whole symbol in parentheses is also underlined it means that the whole $(i, j)$ plane is part of $\mathscr{B}$. The number of different coordinate sections occurring in codimension $K$ is $\frac{1}{2}K(K-1)$, which increases rapidly with $K$. Therefore table 2 stops arbitrarily at codimension 7, giving 21 sections.

The sections displaying curves of type $(p, q)$ are derived from the equations

$$i = K + 2 - q \qquad j = K + 2 - p.$$

Note that if we had chosen to number the control variables in the opposite sense, the $(i, j)$ values in table 2 would have become almost trivial, but then the rule for including tangent lines and planes would have become more complicated, giving no net simplification. Note also that $c_i$, the first numbered in the pair $(i, j)$, *always* corresponds to $-x$, and $c_j$ to $+y$.

As an illustration we consider the butterfly catastrophe $A_5$, which is the first to exhibit a plane of bifurcation points, the (4, 3) plane. The six singular sections are plotted in figures 2($a$)–($f$). They may be compared with the singular sections of the complete unfoldings of the butterfly shown in figures 8($a$), ($b$), ($d$), ($c$), ($e$), ($f$) respectively of Nye and Thorndike (1980). Our figures 2($a$) and 2($d$) may also be compared with the singular sections of figures 8($a$) and 9($a$) of Woodcock and Poston (1974). They are identical except that neither of the other pairs of authors shows the tangent line in the (3, 2) section, our figure 2($d$), nor mentions the tangent plane in the (4, 3) section.

In figure 2 we show the type of catastrophe occurring on each branch of the bifurcation set, and whether or not it occurs at $s = 0$. These figures are self-consistent, in that any particular axis always displays the same catastrophe points, irrespective of which plane it appears in. This suggests that our inclusion of the tangent line in the (3, 2) section is correct.

## 5. Discussion of the classification

We have classified the singular plane coordinate sections of the whole sequence of cuspoid bifurcation sets, and displayed the beginning of the classification in table 2. These sections display one of eight types of singularity, which are equivalent in a weaker sense than the equivalence up to diffeomorphism used to classify the cuspoids themselves.

In all but (2, 1) sections the bifurcation set $\mathscr{B}$ is garnished with a tangent line. In all $(i, j)$ sections with $j \geqslant 3$ the whole $(i, j)$ plane is part of $\mathscr{B}$, and the curves which we are classifying are curves of additional degeneracy. Only cusps and bends ever appear alone, because only cusps and bends can occur in (2, 1) sections. We see this as follows. Since $p = K + 2 - j$, $q = K + 2 - i$, then for $i = 2, j = 1$ we have $p = K + 1$, $q = K$. If $K$ is

**Table 2.** The sections $(i, j)$ displaying a particular curve type, $\underline{i}$ means that the $c_i$ axis forms a tangent line, $\underline{(i, j)}$ means that the whole $(i, j)$ plane is part of the bifurcation set.

| | | $p$ | 3 | 4 | 4 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 7 | 7 | 7 | 7 | 7 | 8 | 8 | 8 | 8 | 8 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $q$ | 2 | 2 | 3 | 2 | 3 | 4 | 2 | 3 | 4 | 5 | 2 | 3 | 4 | 5 | 6 | 2 | 3 | 4 | 5 | 6 | 7 |
| Curve type | | | C3/2 | E2 | B4/3 | C5/2 | K5/3 | C5/4 | E3 | M2 | E3/2 | B6/5 | C7/2 | K7/5 | C7/4 | K7/3 | C7/6 | E4 | B8/3 | E2 | B8/5 | E4/3 | B8/7 |
| Name | | $K$ | | | | | | | | | | | | | | | | | | | | | |
| Cusp | $A_3$ | 2 | $(2, 1)$ | | | | | | | | | | | | | | | | | | | | |
| Swallowtail | $A_4$ | 3 | $(\underline{3}, 2)$ | $(\underline{3}, 1)$ | $(2, 1)$ | | | | | | | | | | | | | | | | | | |
| Butterfly | $A_5$ | 4 | $\underline{(\underline{4}, 3)}$ | $(\underline{4}, 2)$ | $(\underline{3}, 2)$ | $(\underline{4}, 1)$ | $(\underline{3}, 1)$ | $(2, 1)$ | | | | | | | | | | | | | | | |
| — | $A_6$ | 5 | $\underline{(\underline{5}, 4)}$ | $\underline{(\underline{5}, 3)}$ | $\underline{(\underline{4}, 3)}$ | $(\underline{5}, 2)$ | $(\underline{4}, 2)$ | $(\underline{3}, 2)$ | $(\underline{5}, 1)$ | $(\underline{4}, 1)$ | $(\underline{3}, 1)$ | $(2, 1)$ | | | | | | | | | | | |
| — | $A_7$ | 6 | $\underline{(\underline{6}, 5)}$ | $\underline{(\underline{6}, 4)}$ | $\underline{(\underline{5}, 4)}$ | $\underline{(\underline{6}, 3)}$ | $\underline{(\underline{5}, 3)}$ | $\underline{(\underline{4}, 3)}$ | $(\underline{6}, 2)$ | $(\underline{5}, 2)$ | $(\underline{4}, 2)$ | $(\underline{3}, 2)$ | $(\underline{6}, 1)$ | $(\underline{5}, 1)$ | $(\underline{4}, 1)$ | $(\underline{3}, 1)$ | $(2, 1)$ | | | | | | |
| — | $A_8$ | 7 | $\underline{(\underline{7}, 6)}$ | $\underline{(\underline{7}, 5)}$ | $\underline{(6, 5)}$ | $\underline{(\underline{7}, 4)}$ | $\underline{(\underline{6}, 4)}$ | $\underline{(\underline{5}, 4)}$ | $\underline{(\underline{7}, 3)}$ | $\underline{(\underline{6}, 3)}$ | $\underline{(\underline{5}, 3)}$ | $\underline{(\underline{4}, 3)}$ | $(\underline{7}, 2)$ | $(\underline{6}, 2)$ | $(\underline{5}, 2)$ | $(\underline{4}, 2)$ | $(\underline{3}, 2)$ | $(\underline{7}, 1)$ | $(\underline{6}, 1)$ | $(\underline{5}, 1)$ | $(\underline{4}, 1)$ | $(\underline{3}, 1)$ | $(2, 1)$ |

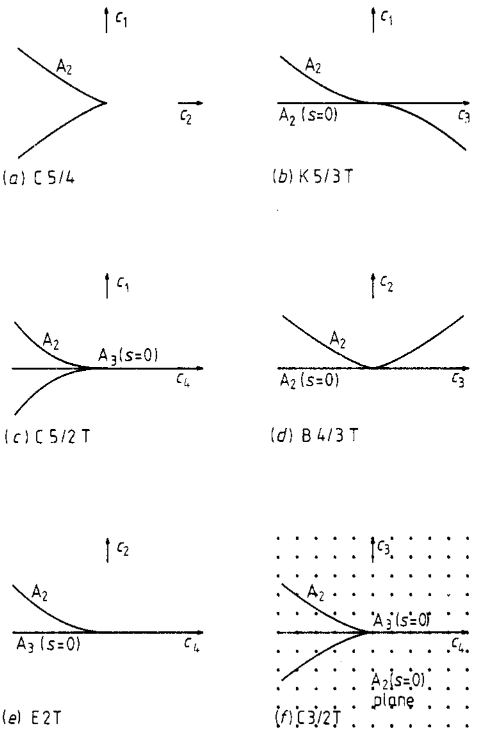


**Figure 2.** The six singular $(i, j)$ sections of the butterfly: (*a*) $(2, 1) = \mathrm{C5/4}$, (*b*) $(3, 1) = \mathrm{K5/3T}$, (*c*) $(4, 1) = \mathrm{C5/2T}$, (*d*) $(3, 2) = \mathrm{B4/3T}$, (*e*) $(4, 2) = \mathrm{E2T}$, (*f*) $(4, 3) = \mathrm{C3/2T}$; the whole plane is also part of the bifurcation set. On the branches of the bifurcation set, the catastrophe type (and whether the degeneracy occurs at $s = 0$) is indicated.

even then $(p, q) =$ (odd, even), which from table 1 gives a cusp, whereas if $K$ is odd then $(p, q) =$ (even, odd), which gives a bend or even monomial. A monomial requires $p/q$ to be integral, where $p/q = (K+1)/K = 1 + 1/K$, which is not possible since we are only considering $K > 1$. Hence isolated monomials, which would not be singular, cannot occur, and neither can isolated kinks or ends: a caustic cannot come to an end as a result of focusing alone!

We look now, in codimension order, at some other specific results of this analysis, and indicate connections and comparisons with other work. The tangent lines first appear in two sections of the swallowtail. In the (3, 2) section, which shows the first occurrence of the CT singularity, the tangent line is due to the (3, 2) plane being tangent to the bifurcation set, whereas in the (3, 1) section, which shows the first occurrence of the ET singularity, it is due to the bifurcation set intersecting the plane. This difference naturally has immense effect on the unfolding of these sections. As mentioned above, the tangent lines are not apparent in the computer-graphical studies of bifurcation sets by Woodcock and Poston (1974) (although these authors are aware of the deficiency—see their remark about non-generic slices on p 8). However, the tangent lines are essential to an understanding of the unfoldings. For example, if the singular (3, 2) section of the butterfly is unfolded along the $c_1$ axis, it unfolds into a cusp and a fold (Woodcock and Poston 1974, p 23). It is obvious how a bend-plus-tangent can unfold in this way, but not at all clear how a bend on its own could (probably it could not!). A similar example occurs for the singular (3, 2) section of $A_6$, this time involving a cusp-plus-tangent (Woodcock and Poston 1974, p 38).

The E branch of the ET singularity in the (3, 1) section of the swallowtail is produced by a line of self-intersection of the bifurcation set, i.e. it is two folds superposed. The continuation, resulting from removing the inequalities, of this semi-algebraic curve is called a *complex whisker* (Poston and Stewart 1976, p 130). On it, complex stationary points coalesce in conjugate pairs. Surrounding a physical swallowtail caustic is a diffraction pattern, the swallowtail diffraction catastrophe (Berry and Upstill 1980). Branching off from the complex whisker are parts of the Stokes and anti-Stokes sets (Wright 1977, 1980) of the diffraction catastrophe, which play an important role in determining the wavefront dislocation structure (Nye and Berry 1974, Wright 1979, Berry 1980).

The B singularity occurs first in the (2, 1) section of the swallowtail which is shown unfolded along the $c_3$ axis on p 18 of Woodcock and Poston (1974). It is important to remember that the slope $\theta(s)$ varies *continuously* through a bend singularity—only the higher derivatives are singular. By contrast, the slope is *discontinuous* at a *corner* singularity, such as is displayed by one of the singular sections of the hyperbolic umbilic. The corner singularity is not displayed by any cuspoid.

BT and KT first appear in the (3, 2) and (3, 1) sections, respectively, of the butterfly, as discussed above.

We have analysed *geometrical* properties of the cuspoid normal forms. However, the normal forms are only convenient representatives of equivalence classes of functions, and it may appear that we have been analysing artifacts of our particular choice of normal forms. Globally this is true, but locally, sufficiently close to their main singularities, all the functions in the equivalence class 'look the same' (they are diffeomorphic) and can be expressed in the same coordinate system. So, for example, tangent lines will always pass (locally) straight through the main catastrophe point, and can never themselves acquire any kind of singularity.

## 6. Applications

For high-codimension catastrophes, the biggest problem in understanding their geometry is lack of space to accommodate the myriad sections one would like to plot. The set of singular coordinate sections constitutes a simple but characteristic skeleton. Although non-generic, they provide useful anchors to tie down the full $K$-dimensional geometry of a bifurcation set. Consideration of how these singular sections could unfold gives clues to the local structure of the bifurcation set, which could be confirmed by a perturbation analysis. The analytical study presented here is complementary to more general numerical studies, since such methods appear generally to have difficulty picking up tangent lines (Upstill, private communication).

One way in which singular sections can occur physically is as badly resolved nearby generic sections, for example, in optical caustics. Berry and Nye (1977) discuss a caustic triple junction, which turns out (Upstill 1979b) to be *organised* by the corank 2 catastrophe ${}^0X_9$, so that the resolution of the 'triple junction' involves an unfolding of ${}^0X_9$. Berry and Nye's photographs, especially figures 5(*d*), (*e*) and (*f*), show features which *locally* look like singular (3, 1) sections of swallowtails. Of course, they cannot really be exactly singular (3, 1) sections—they will be generic sections near this section—but *in practice* the blurring due to diffraction makes all such nearby sections look much the same. By recognising this fact, Berry and Nye were able to draw suitably generic explanatory sketches. In fact, the singular (3, 1) section of the swallowtail on its own gives a triple junction with a rather special geometry!

Singular sections can also arise physically as a result of constraints on a system such as symmetries. If an optical system produces a caustic which is a singular section of a catastrophe, then a generic perturbation will change the topology (diffeotype) of the caustic. Experience suggests that the caustic is likely to be a singular *coordinate* section, especially if it is a result of symmetry. By recognising this section, and using table 2, one can *predict* how the caustic can unfold when the constraint is removed (assuming the catastrophe responsible to be a cuspoid!). For example, an isolated C5/4 cusp can *only* arise from a (2, 1) section of a butterfly (a fact made use of by Wright (1981) in the analysis of a virtual caustic). Most sections, however, may arise from more than one catastrophe. This approach was fruitfully applied to quantum scattering by Berry (1975) to predict a hyperbolic umbilic unfolding from its singular section. Berry then confirmed this both analytically and in an optical analogue experiment.

A particular example of a constraint is if the generating function of a cuspoid must be odd in $s$, as might be required to make the associated diffraction catastrophe real. This constraint is satisfied by cuspoids with odd codimension in hypersections in which all even control variables $c_{2m}$ are zero. Such real hypersections of diffraction catastrophes can occur as 'Wigner catastrophes'—Wigner functions in the phase space of a semiclassical system—which project into normal diffraction catastrophes in the quantum wavefunctions (Berry 1977a, Berry and Wright 1980). Hence any of the sections we have considered, for which $K$, $i$ and $j$ are all odd, could occur as sections of a 'Wigner caustic', part of which forms the classical phase-space manifold. The simplest example, the (3, 1) section of the swallowtail, is discussed in detail in the above references. However, for higher catastrophes Wigner caustics can be more general than our singular coordinate sections.

It should be possible to extend this analysis to the umbilics, in fact, the more complicated the catastrophe the more useful this analysis is, since it is more difficult to do anything else! There are sequences of umbilics in Arnol'd's (1975) classification (see also Berry 1977b), the simplest of which are the conic umbilics $D_j$ ($j \geqslant 4$), whose generating functions take on a general form as do the cuspoids. But there are also a lot of umbilics which would have to be treated individually!

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